A SHARP BOUND FOR THE STEIN-WAINGER OSCILLATORY INTEGRAL

IOANNIS R. PARISSIS

Abstract. Let $\mathcal{P}_d$ denote the space of all real polynomials of degree at most d. It is an old result of Stein and Wainger [4] that

$$\sup_{P\in\mathcal{P}_d}\left|p.v.\int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right|\leq C_d$$

for some constant C_d depending only on d. On the other hand, Carbery, Wainger and Wright in [2] claim that the true order of magnitude of the above principal value integral is $\log d$. We prove that

$$\sup_{P\in\mathcal{P}_d}\left|p.v.\int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right|\sim \log d.$$

1. Introduction

Let $\mathcal{P}_d$ be the vector space of all real polynomials of degree at most d in $\mathbb{R}$. For $P\in\mathcal{P}_d$ we consider the principal value integral

$$I(P)=\left|p.v.\int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t}\right|.$$

We wish to estimate the quantity $I(P)$ by a constant $C(d)$ depending only on the degree of the polynomial d. This amounts to estimating the integral

$$I_{(\epsilon,R)}(P)=\left|\int_{\epsilon\leq|t|\leq R} e^{iP(t)}\frac{dt}{t}\right|$$

by some constant $C(d)$ independent of ϵ, R and P.

This problem is quite old and in fact has been answered some thirty years ago by Stein and Wainger in [4] and [6]. They showed that the quantity $I(P)$ is bounded by a constant C_d depending only on d. Their proof is very simple and uses a combination of induction and Van der Corput's lemma. Let us recall the latter since we'll also be using it in what follows.

2000 *Mathematics Subject Classification.* Primary 42A50; Secondary 42A45.

Proposition 1.1 (van der Corput). *Let $\phi : [a,b] \to \mathbb{R}$ be a C^k function and suppose that $|\phi^{(k)}(t)| \geq 1$ for some $k \geq 1$ and all $t \in [a,b]$. If $k = 1$ suppose in addition that ϕ' is monotonic. Then, for every $\lambda \in \mathbb{R}$,*

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq \frac{Ck}{|\lambda|^{\frac{1}{k}}}$$

where C is an absolute constant independent of a,b,k and ϕ.

For a proof of this very well known result with Ck replaced by C_k see for example [3]. A proof that the constant C_k can be taken to be linear in k can be found in [1].

On the other hand, Carbery, Wainger and Wright have conjectured in [2] that the true order of magnitude of the principal value integral is $\log d$. The main result of this paper is the proof of this conjecture. This is the content of:

Theorem. *There exist two absolute positive constants c_1 and c_2 such that*

$$c_1 \log d \leq \sup_{P \in \mathcal{P}_d} \left| p.v. \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c_2 \log d.$$

Remark 1.2. Suppose that K is a $-n$ homogeneous function on $\mathbb{R}^n$, odd and integrable on the unit sphere. Then, by the one-dimensional result, we trivially get that there is an absolute positive constant c, such that:

$$\left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \|K\|_{L^1(S^{n-1})} \log d,$$

for every polynomial P on $\mathbb{R}^n$, of degree at most d.

Notation. We will use the letter c to denote an absolute positive constant which might change even in the same line of text. Also, the notation $A \sim B$ means that there exist absolute positive constants c_1 and c_2 such that $c_1 B \leq A \leq c_2 B$.

2. Aknowledgements

I would like to thank James Wright for bringing this problem to our attention and for many helpful discussions. I would also like to thank Mihalis Papadimitrakis from the University of Crete, my thesis supervisor, for his constant support.

3. The lower bound in the Theorem

In this section we will construct a real polynomial P of degree at most d such that the inequality

$$I(P) = \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \geq c \log d \tag{3.1}$$

holds. The general plan of the construction is as follows. We will first construct a *function* f (which will not be a polynomial) such that $I(f) \geq c \log n$. We will then construct a polynomial P of degree $d = 2n^2 - 1$ that approximates the function f in a way that $|I(f) - I(P)|$ is small (small means $o(\log n)$ here). Since $\log n \sim \log d$ this will yield our result.

Lemma 3.1. *For n a large positive integer, let $f(t)$ be the continuous function which is equal to 1 for $\frac{1}{n} \leq t \leq 1-\frac{1}{n}$, equal to -1 for $-1+\frac{1}{n} \leq t \leq -\frac{1}{n}$, equal to 0 for $|t| \geq 1$ and linear in each interval $[-1,-1+\frac{1}{n}]$, $[-\frac{1}{n},\frac{1}{n}]$ and $[1-\frac{1}{n},1]$. Then,*

$$I(f) = \left| p.v. \int_{\mathbb{R}} e^{if(t)} \frac{dt}{t} \right| \geq c \log n. \tag{3.2}$$

Proof. The proof is more or less straightforward.

$$\begin{aligned} I(f) &= 2\Big| \int_0^1 \frac{\sin f(t)}{t} dt \Big| \\ &\geq 2\Big| \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\sin f(t)}{t} dt \Big| - 2\Big| \int_0^{\frac{1}{n}} \frac{\sin f(t)}{t} dt \Big| - 2\Big| \int_{1-\frac{1}{n}}^1 \frac{\sin f(t)}{t} dt \Big| \\ &\geq 2 \sin 1 \log(n-1) - 2\int_0^{\frac{1}{n}} \frac{f(t)}{t} dt - 2\int_{1-\frac{1}{n}}^1 \frac{f(t)}{t} dt \\ &= 2 \sin 1 \log(n-1) - 2 - 2n \log \frac{n}{n-1} + 2 \\ &\geq 2 \sin 1 \log(n-1) - 4 \quad \geq \quad c \log n. \end{aligned}$$

□

We now want to construct a polynomial which approximates the function f. We will do so by convolving the function f with a "polynomial approximation to the identity". To be more specific, for $k \in \mathbb{N}$ and $x \in \mathbb{R}$ define the function

$$\phi_k(x) = c_k \left(1 - \frac{x^2}{4}\right)^{k^2} \tag{3.3}$$

where the constant c_k is defined by means of the normalization

$$\int_{-2}^{2} \phi_k(x) dx = 1. \tag{3.4}$$

Observe that

$$1 = c_k \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right)^{k^2} dx = 4c_k \int_0^1 (1-x^2)^{k^2} dx = 2c_k B\left(\frac{1}{2}, k^2+1\right),$$

where $B(\cdot,\cdot)$ is the beta function. Using standard estimates for the beta function we see that $c_k \sim k$.

Define, next, the functions P_k in $\mathbb{R}$ as

$$P_k(t) = \int_{-1}^{1} f(x)\phi_k(t-x) dx, \tag{3.5}$$

where f is the function of Lemma 3.1. It is clear that the functions P_k are *polynomials* of degree at most $2k^2$. The following lemma deals with some technical issues concerning the polynomials P_k.

Lemma 3.2. *Let P_k be defined as in (3.5) above.*
(i) P_k is an odd polynomial of degree $2k^2-1$ with leading coefficient

$$a_k = (-1)^{k^2+1}\frac{2c_k k^2}{4^{k^2}}\Big(1-\frac{1}{n}\Big).$$

That is

$$P_k(t) = a_k t^{2k^2-1} + \cdots .$$

(ii) As a consequence of (i) we have for all t

$$|P_k^{(2k^2-1)}(t)| \geq c(2k^2-1)!\frac{k^3}{4^{k^2}}.$$

(iii) For $t \in [-1,1]$ we have

$$P_k(t) = \int_0^2 \big(f(t+x)+f(t-x)\big)\phi_k(x)dx.$$

Proof. (i) Using (3.5) we have

$$\begin{aligned} P_k(-t) &= \int_{-1}^1 f(x)\phi_k(-t-x)dx = \int_{-1}^1 f(x)\phi_k(t+x)dx \\ &= \int_{-1}^1 f(-x)\phi_k(t-x)dx = -P_k(t). \end{aligned}$$

Next, from (3.5) we have that

$$\begin{aligned} P_k(t) &= c_k \int_{-1}^1 f(x) \sum_{m=0}^{k^2} \binom{k^2}{m}\Big(-\frac{(t-x)^2}{4}\Big)^m dx \\ &= c_k \sum_{m=0}^{k^2}\binom{k^2}{m}\frac{(-1)^m}{4^m}\int_{-1}^1 f(x)(t-x)^{2m}dx \\ &= c_k \frac{(-1)^{k^2}}{4^{k^2}}\int_{-1}^1 f(x)(x-t)^{2k^2}dx \\ &+ c_k \sum_{m=0}^{k^2-1}\binom{k^2}{m}\frac{(-1)^m}{4^m}\int_{-1}^1 f(x)(t-x)^{2m}dx. \end{aligned}$$

It is now easy to see that the two highest order terms come from the first summand in the above formula. Therefore,

$$\begin{aligned} P_k(t) &= c_k\frac{(-1)^{3k^2}}{4^{k^2}}\int_{-1}^1 f(x)dx\ t^{2k^2} - c_k\frac{(-1)^{k^2}2k^2}{4^{k^2}}\int_{-1}^1 f(x)xdx\ t^{2k^2-1} + \cdots \\ &= (-1)^{k^2+1}\frac{2c_k k^2}{4^{k^2}}\Big(1-\frac{1}{n}\Big)t^{2k^2-1} + \cdots . \end{aligned}$$

(ii) We just use the result of (i) and that $c_k \sim k$.

(iii) Fix a $t \in [-1,1]$. Then,

$$\begin{aligned}\int_{-2}^{2} f(t-x)\phi_k(x)dx &= \int_{\mathbb{R}} f(t-x)\phi_k(x)\chi_{[-2,2]}(x)dx \\ &= \int_{-1}^{1} f(x)\phi_k(t-x)\chi_{[-2,2]}(t-x)dx \\ &= \int_{-1}^{1} f(x)\phi_k(t-x)dx \\ &= P_k(t).\end{aligned}$$

However, since ϕ_k is even,

$$P_k(t) = \int_{-2}^{2} f(t-x)\phi_k(x)dx = \int_{0}^{2} \big(f(t+x)+f(t-x)\big)\phi_k(x)dx.$$

$\square$

We are now ready to prove the lower bound for $I(P)$.

Proposition 3.3. *Let P_n be the polynomial defined in (3.5) where n is the large positive integer used to define the function f in Lemma 3.1. Then P_n is a polynomial of degree $d = 2n^2 - 1$ and*

$$I(P_n) = \left| p.v. \int_{\mathbb{R}} e^{iP_n(t)} \frac{dt}{t} \right| \geq c \log d.$$

Proof. Since P_n is odd,

$$I(P_n) = 2\left| \int_0^{+\infty} \frac{\sin P_n(t)}{t} dt \right|,$$

and it suffices to show that for all $R \geq 1$

$$\left| \int_0^R \frac{\sin P_n(t)}{t} dt \right| \geq c \log d \sim c \log n. \tag{3.6}$$

By part (ii) of Lemma 3.2 and a standard application of Proposition 1.1 (Van der Corput) we see that

$$\left| \int_1^R \frac{\sin P_n(t)}{t} dt \right| \leq c$$

for all $R \geq 1$. As a result, the proof will be complete if we show that

$$I_1(P_n) = \left| \int_0^1 \frac{\sin P_n(t)}{t} dt \right| \geq c \log n. \tag{3.7}$$

Using Lemma 3.1 and the triangle inequality we get

$$I_1(P_n) \geq c \log n - |I_1(P_n) - I(f)| \tag{3.8}$$

and, in order to show (3.7), it suffices to show that

$$|I_1(P_n) - I(f)| = o(\log n). \tag{3.9}$$

We have that

$$\begin{aligned} |I_1(P_n) - I(f)| &= \left| \int_0^1 \frac{\sin P_n(t) - \sin f(t)}{t} dt \right| \\ &\le \int_0^1 \frac{|P_n(t) - f(t)|}{t} dt. \end{aligned}$$

Using part (iii) of Lemma 3.2 and (3.4), we get

$$|P_n(t) - f(t)| \le \int_0^2 |f(t+x) + f(t-x) - 2f(t)| \phi_n(x) dx$$

for $0 \le t \le 1$. Hence

$$|I_1(P_n) - I(f)| \le \int_0^2 \int_0^1 \frac{|f(t+x) + f(t-x) - 2f(t)|}{t} \, dt \, \phi_n(x) dx.$$

Now, the desired result, condition (3.9), is the content of the following lemma. □

Lemma 3.4. *Let* $A(x,t) = |f(t+x) + f(t-x) - 2f(t)|$*. Then,*

$$\int_0^2 \int_0^1 \frac{A(x,t)}{t} \, dt \, \phi_n(x) dx = o(\log n).$$

Proof. Firstly, it is not difficult to establish that

$$A(x,t) \le 4 \min(nx, nt, 1) \tag{3.10}$$

$$A(x,t) = 0, \qquad \text{when } \frac{1}{n} \le t - x \le t + x \le 1 - \frac{1}{n}. \tag{3.11}$$

Indeed,

$$\begin{aligned} A(x,t) &\le |f(t+x) - f(t)| + |f(t-x) - f(t)| \\ &\le nx + nx \le 2nx. \end{aligned}$$

On the other hand,

$$\begin{aligned} A(x,t) &= |f(t+x) - f(x) + f(t-x) - f(-x) - 2f(t)| \\ &\le |f(t+x) - f(x)| + |f(t-x) - f(-x)| + 2|f(t)| \\ &\le nt + nt + 2nt = 4nt. \end{aligned}$$

Inequality (3.10) now follows by the fact that $|f|$ is bounded by 1 and (3.11) is trivial to prove.

We split the integral $\int_0^2 \int_0^1 \cdots dtdx$ into seven integrals:

$$\int_0^2 \int_{\frac{1}{2}}^1 \cdots dtdx + \int_0^{\frac{1}{n}} \int_0^x \cdots dtdx + \int_{\frac{1}{n}}^2 \int_0^{\frac{1}{n}} \cdots dtdx + \int_0^{\frac{1}{n}} \int_x^{x+\frac{1}{n}} \cdots dtdx$$
$$+ \int_0^{\frac{1}{2}-\frac{1}{n}} \int_{x+\frac{1}{n}}^{\frac{1}{2}} \cdots dtdx + \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \int_{\frac{1}{n}}^{x+\frac{1}{n}} \cdots dtdx + \int_{\frac{1}{2}-\frac{1}{n}}^2 \int_{\frac{1}{n}}^{\frac{1}{2}} \cdots dtdx.$$

We estimate each of the seven integrals separately.

$$\int_0^2 \int_{\frac{1}{2}}^1 \frac{A(x,t)}{t} dt \phi_n(x) dx \le 4 \log 2 \int_0^2 \phi_n(x) dx = 2 \log 2.$$

$$\begin{aligned}\int_0^{\frac1n}\int_0^x \frac{A(x,t)}{t}dt\phi_n(x)dx &\le \int_0^{\frac1n}\int_0^x \frac{4nt}{t}dt\phi_n(x)dx\\ &= \int_0^{\frac1n} 4nx\phi_n(x)dx \;\le\; 2.\end{aligned}$$

$$\begin{aligned}\int_{\frac1n}^2\int_0^{\frac1n} \frac{A(x,t)}{t}dt\phi_n(x)dx &\le \int_{\frac1n}^2\int_0^{\frac1n} \frac{4nt}{t}dt\phi_n(x)dx\\ &= \int_0^{\frac1n} 4\phi_n(x)dx \;\le\; 2.\end{aligned}$$

$$\begin{aligned}\int_0^{\frac1n}\int_x^{x+\frac1n} \frac{A(x,t)}{t}dt\phi_n(x)dx &\le \int_0^{\frac1n}\int_x^{x+\frac1n} \frac{4nx}{t}dt\phi_n(x)dx\\ &= \int_0^{\frac1n} 4nx\log\left(1+\frac{1}{nx}\right)\phi_n(x)dx \;\le\; 2.\end{aligned}$$

For $\int_0^{\frac12-\frac1n}\int_{x+\frac1n}^{\frac12}$ we have $\frac1n \le t-x \le t+x \le 1-\frac1n$ and, by (3.11), $A(x,t)=0$. Hence

$$\int_0^{\frac12-\frac1n}\int_{x+\frac1n}^{\frac12}\frac{A(x,t)}{t}dt\phi_n(x)dx = 0.$$

Next

$$\begin{aligned}\int_{\frac1n}^{\frac12-\frac1n}\int_{\frac1n}^{x+\frac1n} \frac{A(x,t)}{t}dt\phi_n(x)dx &\le \int_{\frac1n}^{\frac12-\frac1n}\int_{\frac1n}^{x+\frac1n} \frac{4}{t}dt\phi_n(x)dx\\ &\le 4\int_{\frac1n}^1 \log(nx+1)\phi_n(x)dx.\end{aligned}$$

Now, fix some $\alpha\in(0,1)$. Write

$$\begin{aligned}\int_{\frac1n}^1 \log(nx+1)\phi_n(x)dx &= \int_{\frac1n}^{\frac{1}{n^\alpha}}\cdots dx + \int_{\frac{1}{n^\alpha}}^1\cdots dx\\ &\le \frac{\log(n^{1-\alpha}+1)}{2} + c_n\log(n+1)\int_{\frac{1}{n^\alpha}}^1\left(1-\frac{x^2}{4}\right)^{n^2}dx\\ &\le \frac{\log(n^{1-\alpha}+1)}{2} + cn\log(n+1)\; e^{-\frac14 n^{2(1-\alpha)}}.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}\frac{\int_{\frac1n}^1 \log(nx+1)\phi_n(x)dx}{\log n}\le\frac{1-\alpha}{2}$$

and, since α is arbitrary in $(0,1)$,

$$\int_{\frac1n}^{\frac12-\frac1n}\int_{\frac1n}^{x+\frac1n}\frac{A(x,t)}{t}dt\phi_n(x)dx = o(\log n).$$

Finally,

$$\begin{aligned}
\int_{\frac{1}{2}-\frac{1}{n}}^{2}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_{\frac{1}{2}-\frac{1}{n}}^{2}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{4}{t}dt\phi_n(x)dx\\
&\leq 4\log\frac{n}{2}\ c_n\int_{\frac{1}{2}-\frac{1}{n}}^{2}\left(1-\frac{x^2}{4}\right)^{n^2}dx\\
&\leq cn\log ne^{-\frac{1}{16}n^2} = o(1).
\end{aligned}$$

□

4. The upper bound in the Theorem

We set

$$K_d=\sup_{P\in\mathcal{P}_d,\epsilon,R}\left|\int_{\epsilon\leq|t|\leq R}e^{iP(t)}\frac{dt}{t}\right|. \tag{4.1}$$

We take any polynomial P, of degree at most d, which we can assume has no constant term, that is, $P(0)=0$. We set $k=[\frac{d}{2}]$ and we write

$$\begin{aligned}
P(t) &= a_1t+a_2t^2+\cdots+a_kt^k+a_{k+1}t^{k+1}+\cdots+a_dt^d\\
&= Q(t)+R(t),
\end{aligned}$$

where $Q(t)=a_1t+a_2t^2+\cdots+a_kt^k$ and $R(t)=a_{k+1}t^{k+1}+\cdots+a_dt^d$. Let $|a_l|=max_{k+1\leq j\leq d}|a_j|$ for some $k+1\leq l\leq d$. By a change of variables in the integral in (4.1) we can assume that $|a_l|=1$ and thus that $|a_j|\leq 1$ for every $k+1\leq j\leq d$. Now split the integral in (4.1) in two parts as follows

$$\begin{aligned}
\left|\int_{\epsilon\leq|t|\leq R}e^{iP(t)}\frac{dt}{t}\right| &\leq \left|\int_{\epsilon\leq|t|\leq 1}e^{iP(t)}\frac{dt}{t}\right|+\left|\int_{1\leq|t|\leq R}e^{iP(t)}\frac{dt}{t}\right|\\
&= I_1+I_2.
\end{aligned} \tag{4.2}$$

For I_1 we have that

$$\begin{aligned}
I_1 &\leq \left|\int_{\epsilon\leq|t|\leq 1}\left[e^{iP(t)}-e^{iQ(t)}\right]\frac{dt}{t}\right|+\left|\int_{\epsilon\leq|t|\leq 1}e^{iQ(t)}\frac{dt}{t}\right|\\
&\leq \int_{\epsilon\leq|t|\leq 1}\left|e^{iP(t)}-e^{iQ(t)}\right|\frac{dt}{t}+K_{[\frac{d}{2}]}\\
&\leq \int_{0\leq|t|\leq 1}\frac{|R(t)|}{t}dt+K_{[\frac{d}{2}]}\\
&\leq 2\sum_{j=k+1}^{d}\frac{|a_j|}{j}+K_{[\frac{d}{2}]} \ \leq \ \sum_{j=k+1}^{d}\frac{1}{j}+K_{[\frac{d}{2}]} \ \leq \ c+K_{[\frac{d}{2}]}.
\end{aligned}$$

For the second integral in (4.2) we have that

$$I_2\leq\left|\int_{1\leq t\leq R}e^{iP(t)}\frac{dt}{t}\right|+\left|\int_{-R\leq t\leq -1}e^{iP(t)}\frac{dt}{t}\right|=I_2^+ + I_2^-.$$

For some $\alpha>0$ to be defined later split I_2^+ into two parts as follows:

$$I_2^+\leq\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}}\frac{dt}{t}+\left|\int_{\{t\in[1,R]:|P'(t)|>\alpha\}}e^{iP(t)}\frac{dt}{t}\right|.$$

Since $\{t \in [1,R] : |P'(t)| > \alpha\}$ consists of at most $O(d)$ intervals where P' is monotonic, using Proposition 1 we get the bound

$$\left| \int_{\{t\in[1,R]:|P'(t)|>\alpha\}} e^{iP(t)} \frac{dt}{t} \right| \le c\frac{d}{\alpha}.$$

For the logarithmic measure of the set $\{t \in [1,+\infty) : |P'(t)| \le \alpha\}$, observe that

$$\begin{aligned}
\int_{\{t\in[1,+\infty):|P'(t)|\le\alpha\}} \frac{dt}{t} &\le \sum_{m=0}^{\infty} \int_{\{t\in[2^m,2^{m+1}]:|P'(t)|\le\alpha\}} \frac{dt}{t} \\
&= \sum_{m=0}^{\infty} \int_{\{2^m t\in[2^m,2^{m+1}]:|P'(2^m t)|\le\alpha\}} \frac{dt}{t} \\
&= \sum_{m=0}^{\infty} \int_{2^m\{t\in[1,2]:|P'(2^m t)|\le\alpha\}} \frac{dt}{t} \\
&= \sum_{m=0}^{\infty} \int_{\{t\in[1,2]:|P'(2^m t)|\le\alpha\}} \frac{dt}{t}.
\end{aligned}$$

We have thus showed that

$$\int_{\{t\in[1,+\infty):|P'(t)|\le\alpha\}} \frac{dt}{t} \le \sum_{m=0}^{\infty} |\{t \in [1,2] : |P'(2^m t)| \le \alpha\}|. \tag{4.3}$$

In order to finish the proof we need a suitable estimate for the sublevel set of a polynomial. This is the content of the following lemma.

Lemma 4.1 (Vinogradov)**.** *Let $h(t) = b_0 + b_1 t + \cdots + b_n t^n$ be a real polynomial of degree n. Then,*

$$|\{t \in [1,2] : |h(t)| \le \alpha\}| \le c\left(\frac{\alpha}{\max_{0\le k\le n} |b_k|}\right)^{\frac{1}{n}}.$$

This Lemma is due to Vinogradov [5]. We postpone the proof of Lemma 4.1 until after the end of the proof of the upper bound.

Consider the polynomial $P'(2^m t)$ with coefficients $ja_j 2^{m(j-1)}$, $1 \le j \le d$. Clearly, $\max_{1\le j\le d} |ja_j 2^{m(j-1)}| \ge |la_l 2^{m(l-1)}| \ge ([\frac{d}{2}] + 1)2^{m[\frac{d}{2}]}$. Using Lemma 4 and (4.3), we get

$$\int_{\{t\in[1,+\infty):|P'(t)|\le\alpha\}} \frac{dt}{t} \le c\alpha^{\frac{1}{d-1}} \sum_{m=0}^{\infty} \left(\frac{1}{([\frac{d}{2}]+1)2^{m[\frac{d}{2}]}}\right)^{\frac{1}{d-1}} \le c\alpha^{\frac{1}{d-1}}.$$

Obviously, a similar estimate holds for I_2^-. Summing up the estimates we get

$$\left| \int_{\epsilon\le|t|\le R} e^{iP(t)} \frac{dt}{t} \right| \le c + c\frac{d}{\alpha} + c\alpha^{\frac{1}{d-1}} + K_{[\frac{d}{2}]}.$$

Optimizing in α we get that

$$\left| \int_{\epsilon\le|t|\le R} e^{iP(t)} \frac{dt}{t} \right| \le c + K_{[\frac{d}{2}]} \tag{4.4}$$

and hence

$$K_d \le c + K_{[\frac{d}{2}]}.$$

In particular we have

$$K_{2^n} \leq c + K_{2^{n-1}}.$$

Using induction on n we get that $K_{2^n} \leq cn$. It is now trivial to show the inequality for general d. Indeed, if $2^{n-1} < d \leq 2^n$ then $K_d \leq K_{2^n} \leq cn \leq c \log d$.

For the sake of completeness we give the proof of Lemma 4.1.

Proof of Lemma 4.1. The set $E_\alpha = \{t \in [1,2] : |h(t)| \leq \alpha\}$ is a union of intervals. We slide them together to form a single interval I of length $|E_\alpha|$ and pick $n+1$ equally spaced points in I. If we slide the intervals back to their original position we end up with $n+1$ points $x_0, x_1, x_2, \ldots, x_n \in E_\alpha$ which satisfy

$$|x_j - x_k| \geq |E_\alpha| \frac{|j-k|}{n}. \tag{4.5}$$

The Lagrange polynomial which interpolates the values $h(x_0)$, $h(x_1)$,..., $h(x_n)$ coincides with $h(x)$:

$$h(x) = \sum_{j=0}^{n} h(x_j) \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}.$$

Therefore we get for the coefficients of h that

$$b_k = \sum_{j=0}^{n} h(x_j) \frac{(-1)^{n-k}\sigma_{n-k}(x_0, \ldots, \hat{x_j}, \ldots, x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

for $k = 0, 1, \ldots, n$. In the above formula $\sigma_{n-k}(x_0, \ldots, \hat{x_j}, \ldots, x_n)$ is the $(n-k)$-th elementary symmetric function of $x_0, \ldots, \hat{x_j}, \ldots, x_n$ where x_j is omitted. Using the estimate $\sigma_{n-k}(x_0, \ldots, \hat{x_j}, \ldots, x_n) \leq \binom{n}{n-k} 2^{n-k}$ together with (4.5) we get that, for every $k = 0, 1, \ldots, n$,

$$\begin{aligned} |b_k| \quad &\leq \quad \binom{n}{n-k} 2^{n-k} n^n \frac{\alpha}{|E_\alpha|^n} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \\ &= \quad \binom{n}{n-k} 2^{2n-k} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n} \quad \leq \quad c \frac{8^n}{\sqrt{n}} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n}, \end{aligned}$$

where we used the estimate $\binom{n}{n-k} \leq \binom{n}{[\frac{n}{2}]} \leq c \frac{2^n}{\sqrt{n}}$. Hence

$$\max_{0 \leq k \leq n} |b_k| \leq c \frac{8^n}{\sqrt{n}} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n}$$

and solving with respect to $|E_\alpha|$ we get

$$|E_\alpha| \leq c \left(\frac{\alpha}{\max_{0 \leq k \leq n} |b_k|} \right)^{\frac{1}{n}}.$$

□

References

1. G. I. Arhipov, A. A. Karacuba, and V. N. Čubarikov, *Trigonometric integrals*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 5, 971–1003, 1197. MR MR552548 (81f:10050)
2. Anthony Carbery, Stephen Wainger, and James Wright, *Personal communication*, 2005.
3. E. M. Stein, *Oscillatory integrals in Fourier analysis*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355. MR MR864375 (88g:42022)

4. Elias M. Stein and Stephen Wainger, *The estimation of an integral arising in multiplier transformations.*, Studia Math. **35** (1970), 101–104. MR MR0265995 (42 #904)
5. Ivan Matveevič Vinogradov, *Selected works*, Springer-Verlag, Berlin, 1985, With a biography by K. K. Mardzhanishvili, Translated from the Russian by Naidu Psv [P. S. V. Naidu], Translation edited by Yu. A. Bakhturin. MR MR807530 (87a:01042)
6. Stephen Wainger, *Averages and singular integrals over lower-dimensional sets*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 357–421. MR MR864376 (89a:42026)

Department of Mathematics, University of Crete, Knossos Avenue 71409, Iraklio–Crete, Greece
E-mail address: ypar@math.uoc.gr